Indian Statistical Institute
Second Semester Exam 2005-2006
B.Math. (Hons.) I Year
Analysis II

Time: 3 hrs — Date: -05-06

The paper carries 54 marks; the maximum you can score is 50.

1. a) Let $f : R^2 \longrightarrow R$ be a continuous function such that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}$ all exist and are continuous. Let $ABCD$ be a rectangle with AB parallel to x-axis. Show that
$$\int_S \frac{\partial^2 f}{\partial x \partial y}(x,y)dx\ dy = f(A) + f(C) - f(B) - f(D)$$
where if $A = (x_1, y_1),\ C = (x_2, y_2)$ with $x_1 < x_2,\ y_1, < y_2$ then $S = \{(x,y)\ :\ x_1 \leq x \leq x_2,\ \ y_1 \leq y \leq y_2\}$ [2]
b) If further $\frac{\partial^2 f}{\partial y \partial x}$ exists and continuous show that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ on R^2. [2]

2. Let $g : R^2 \longrightarrow R$ be any C^2 function such that $g(\underset{\sim}{0}) = 0, (\nabla g)(\underset{\sim}{0}) = 0$. Also let
$$G = \begin{pmatrix} \frac{\partial^2 g}{\partial x^2}(\underset{\sim}{0}) & \frac{\partial^2 g}{\partial x \partial y}(\underset{\sim}{0}) \\ \frac{\partial^2 g}{\partial y \partial x}(\underset{\sim}{0}) & \frac{\partial^2 g}{\partial y^2}(\underset{\sim}{0}) \end{pmatrix}$$
satisfy $\langle G\underset{\sim}{y}, \underset{\sim}{y}\rangle\ > 0$ for $\underset{\sim}{y} \neq \underset{\sim}{0}$. Show that $\underset{\sim}{0}$ is local minima for g. [6]

3. Let $g : (X,d) \longrightarrow (Y,m)$ be any function between metric spaces. Show that g is uniformly continuous iff whenever $d(x_n,\ a_n) \to 0$ one gets $m(g(x_n),\ g(a_n)) \longrightarrow 0$. [3]

4. Let $f(x) = x^2$ for real x.

   a) Let $A = \bigcup\limits_{n=2}^{\infty}[n,\ n + \frac{1}{n}]$. Show that $f : A \to R$ is not uniformly continuous. [1]

   b) Let $B = \bigcup\limits_{n=3}^{\infty}[n,\ n + \frac{1}{n^2}]$. Show that $f : B \to R$ is uniformly continuous. [4]

   c) Give an example of a metric space (X,d), uniformly continuous functions $f, g : X \to R$ such that the product fg is not uniformly continuous. [2]

5. Let A be any connected and compact subset of R with atleast two points. Show that $A = [a, b]$ for some $a < b$. [3]

6. a) Let A be a dense subset of (X, d) i.e every x in$X$ is a limit of a sequence from A. If A is connected, show that X is connected. [3]

   b) Let $X = \{(x, \sin(\frac{1}{x})) \ : \ 0 < x \leq 1\} \cup \{(0, y) \ : \ -1 \leq y \leq 1\}$ with $A = \{(x, \ \sin(\frac{1}{x})) : 0 < x < 1\}$. Show that A is connected and X is connected. [3]

7. (X, d) is said to be a completion of A if A is a dense subset of X and X is complete. Find completion of (i) rationals of R. (ii) $\{(x, y) : 0 < x^2 + y^2 < 1\}$ [2]

8. Let $(R, \ D)$ be the reals with discrete metric D given by $D(x, y) = 1$ if $x \neq y, 0$ if $x = y$.

   a) Show that $\{x\}$ is open for each x.

   b) if A is a connected nonempty subset of (R, D), then A has exactly one element.

   c) If A is a nonempty compact subset of (R, D), then A is a finite set.

   d) Give an example of closed and bounded noncompact subset.

   e) Let $g : (R, D) \longrightarrow (R, u)$, where u is the usual metric on R be given by $g(x) = x$. (i) Show that g is a continuous function. Find subsets A, B of (R, D) such that (ii) A is open but $g(A)$ is not open (iii) B is closed but $g(B)$ is not closed. [7]

9. Let (X, d) be a metric space, $f, g \ : \ (X, d) \rightarrow R$ uniformly continuous. Assume that g and f are bounded. Then the product fg is uniformly continuous. [5]

10. Let (X, d) be a metric space. If $A_1, \ A_2$ are compact subsets of (X, d), show that

    a) $A_1 \cup A_2$ is compact. [4]

    b) $A_1 \cap A_2$ is a closed subset of A_1. [1]

    c) $A_1 \cap A_2$ is compact. [2]

11. a) Show that $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is not a compact subset of R. [1]

    b) Show that $B = \{1, 2, 3, \ldots\}$ is not a compact subset of R. [1]

    c) Let $d(x, y) = \frac{|x-y|}{1+|x-y|}$ for $x, y \in R$. Is A compact set in (R, d) ? Is B compact set in (R, d)? Justify your answer. [2]